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Zeros of the Hypergeometric Polynomials $F(-n, b; -2n; z)$

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number of real zeros in the intervals $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$. For $b > 0$ we obtain the equation of the Cassini curve which the zeros of $w^n F(-n, b; -2n; 1/w)$ approach as $n \to \infty$ and thereby prove a special case of a conjecture made by Martínez-Finkelshtein, Martínez-González, and Orive. We also present some numerical evidence linking the zeros of F with more general Cassini curves.

Key Words: zeros of hypergeometric polynomials; asymptotics of zeros; Jacobi polynomials; Cassini curves.

1. INTRODUCTION

The hypergeometric function is defined by

$$F(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad |z| < 1,$$

where

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$$

is Pochhammer's symbol. If $a = -n$, the series terminates and reduces to a polynomial of degree n, called a hypergeometric polynomial. The question of the location of the zeros of the polynomial $F(-n, b; c; z)$ for different values of the free parameters b and c arises in a natural way and has been the subject of investigation for special cases of real b and c in [2–4]. A

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common feature of the types of hypergeometric polynomials investigated for their zero distribution is that they all admit quadratic transformations. The necessary and sufficient condition, due to Kummer, for a hypergeometric function $F(a, b; c; z)$ to have a quadratic transformation is that the numbers $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ should be such that either one of them equals $\frac{1}{2}$ or two of them are equal. This gives rise to a class of twelve functions, a complete list being (cf. [12, p. 124])

$$\begin{array}{lll}
F\left(a, b; \dfrac{1}{2}; z\right) & F\left(a, a+\dfrac{1}{2}; c; z\right) & F\left(a, b; a+b+\dfrac{1}{2}; z\right) \\[2ex]
F\left(a, b; \dfrac{3}{2}; z\right) & F\left(a, a-\dfrac{1}{2}; c; z\right) & F\left(a, b; a+b-\dfrac{1}{2}; z\right) \\[2ex]
F(a, b; 2a; z) & F(a, b; b-a+1; z) & F(a, 1-a; c; z) \\[2ex]
F(a, b; 2b; z) & F(a, b; a-b+1; z) & F\left(a, b; \dfrac{a+b+1}{2}; z\right).
\end{array}$$

It is clear from the definition of F that $F(a, b; c; z) = F(b, a; c; z)$ and therefore, in the table given above, the first two cases in the second column are not different, nor are the final two cases in the first column and the second column. However, it is apparent that if we fix $a = -n$ and allow b to vary, the symmetry between a and b is lost and all the cases are distinct. We have shown in [4] that when $a = -n$, there are identities linking the hypergeometric polynomials across each row of the table. It is therefore sufficient in analyzing the zeros of polynomials in this quadratic class to restrict our attention to those occurring in the first column only. Three of the four, namely $F(-n, b; \frac{1}{2}; z)$, $F(-n, b; \frac{3}{2}; z)$, and $F(-n, b; 2b; z)$, can be expressed in terms of ultraspherical (Gegenbauer) polynomials and the location of their zeros as b ranges through real values is described in [4]. The remaining polynomial $F(-n, b; -2n; z)$ does not appear to have a connection with ultraspherical polynomials and finding the nature and location of its zeros is our topic of investigation here.

We have from the definition that

$$\begin{aligned}
F(-n, b; -2n; z) &= 1 + \sum_{j=1}^{\infty} \frac{(-n)_j\,(b)_j}{(-2n)_j}\,\frac{z^j}{j!} \\
&= 1 + \sum_{j=1}^{n} \frac{n!(2n-j)!}{(n-j)!\,(2n)!}\,(b)_j\,\frac{z^j}{j!} \qquad (1.1)
\end{aligned}$$

since

$$(-n)_j = \begin{cases} (-1)^j\, n!/(n-j)!, & \text{for} \quad 1 \leqslant j \leqslant n, \\ 0 & j \geqslant n+1. \end{cases}$$

We want to investigate the dependence of the zeros of this function on b for integers $n \geqslant 1$. First, we observe that F is a polynomial of degree n if $b \notin \{-n+1, -n+2, ..., 0\}$, whereas the degree drops for $b \in \{-n+1, -n+2, ..., 0\}$. However, throughout our discussion we shall regard $F(-n, b; -2n; z)$ as a polynomial of degree n and will consider $\lim_{b \to -n+k} F(-n, b; -2n; z)$, $k = 1, ..., n$, rather than the polynomial of reduced degree $n-k$. This ensures that the zeros of F depend continuously on b. In fact, as b approaches $-n+k$, $k = 1, ..., n$, k zeros of $F(-n, b; -2n; z)$ tend to infinity thereby reflecting the reduced degree of F.

In Section 2 we recall that $F(-n, b; -2n; z)$ can be expressed in terms of Jacobi polynomials $P_n^{(-\alpha, \alpha)}$. Hilbert's result on the numbers of real zeros of the Jacobi polynomials in various intervals is stated for the particular case $P_n^{(-\alpha, \alpha)}$. This is used in Section 3 to specify the number of real zeros of F in the interval $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$.

In Section 4 we are going to show that the zeros of $F(-n, b; -2n; z)$ for $b > 0$ approach a fixed curve independent of b in the complex plane as $n \to \infty$. For the formulation it is easier to replace z by $\frac{1}{w}$. We will show that the zeros of

$$w^n F\left(-n, b; -2n; \frac{1}{w}\right)$$

approach the Cassini curve

$$|(2w-1)^2 - 1| = 1 \tag{1.2}$$

as $n \to \infty$.

It is interesting to note that our result, suitably transformed into the corresponding Jacobi polynomial, proves a special case of the conjecture made by Martínez-Finkelshtein *et al.* (cf. [6, Conjecture 1]). In [6], the asymptotic zero distribution of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ is investigated as $\alpha_n/n \to A$ and $\beta_n/n \to B$ as $n \to \infty$, and our result deals with the case $A = -1$ and $B = 1$.

Finally, in Section 5 we present some graphs which indicate that the zeros of $w^n F(-n, b; -2n; \frac{1}{w})$ have a regular pattern and are very close to suitable Cassini curves $|(w-a)^2 - c^2| = d^2$.

2. CONNECTION WITH JACOBI POLYNOMIALS AND HILBERT'S RESULT

The hypergeometric polynomials $F(-n, b; -2n; z)$ can be expressed in terms of the Jacobi polynomial $P_n^{(\alpha, \beta)}(w)$ via the equation (cf. [8, p. 464, Eq. (142)])

$$F(-n, b; -2n; z) = \frac{n!\, z^n}{(-2n)_n} P_n^{(\alpha, \beta)}\left(1 - \frac{2}{z}\right), \tag{2.1}$$

where $\alpha = -b - n$ and $\beta = -\alpha = b + n$. Since (cf. [9, p. 256])

$$P_n^{(\alpha, -\alpha)}(-z) = (-1)^n P_n^{(-\alpha, \alpha)}(z), \tag{2.2}$$

we need only consider either $\alpha > 0$ or $\alpha < 0$ when investigating the zeros of $P_n^{(\alpha, -\alpha)}(w)$, as corresponding results for the other case will follow from the symmetry relation (2.2). Using Klein's symbol

$$E(u) = \begin{cases} 0 & \text{if } \; u \leqslant 0 \\ [u] & \text{if } \; u > 0 \text{ is non-integral} \\ u - 1 & \text{if } \; u = 1, 2, ..., \end{cases} \tag{2.3}$$

Hilbert's result (cf. [11, p. 145, Theorem 6.72]) for the numbers of real zeros of $P_n^{(\alpha, \beta)}(z)$ in the intervals $(-1, 1)$, $(-\infty, -1)$, and $(1, \infty)$ simplifies to the following theorem in our case where $\alpha = -\beta$, α is real and negative.

Theorem 2.1. *Let α be an arbitrary real negative number, $\alpha \neq -1, -2, ..., -n$. Then the numbers of the zeros of $P_n^{(\alpha, -\alpha)}(w)$ in $(-1, 1)$, $(-\infty, -1)$, and $(1, \infty)$ respectively are*

$$N_1 = \begin{cases} 2[(X+1)/2] & \text{if } \; (-1)^n \binom{n+\alpha}{n} > 0 \\[2ex] 2[X/2] + 1 & \text{if } \; (-1)^n \binom{n+\alpha}{n} < 0, \end{cases} \tag{2.4}$$

$$N_2 = 0, \tag{2.5}$$

$$N_3 = \begin{cases} 0 & \text{if } \; \binom{n+\alpha}{n} > 0 \\[2ex] 1 & \text{if } \; \binom{n+\alpha}{n} < 0, \end{cases} \tag{2.6}$$

where

$$X = E(n+1+\alpha). \tag{2.7}$$

Remark. The proof of Theorem 2.1 is contained in the proof of the more general result which is given in detail in [11, pp. 144–149]. It is simply the special case $\beta = -\alpha$ with α negative and real and $\alpha \notin \{-1, -2, ..., -n\}$.

3. NUMBER AND LOCATION OF REAL ZEROS

THEOREM 3.1. *Let $F(-n, b; -2n; z)$ be defined by (1.1) with b real and $n \in \mathbb{N}$.*

(i) *For $b > 0$, the n zeros of F are all non-real if n is even whereas if n is odd, there is exactly one real negative zero and the remaining $(n-1)$ zeros of F are all non-real. All non-real zeros of F occur in complex conjugate pairs.*

(ii) *For $-n < b < 0$, if $-k < b < -k+1$, $k \in \{1, ..., n\}$, $F(-n, b; -2n; z)$ has k real zeros in the interval $(1, \infty)$. In addition, if $(n-k)$ is even, F has $(n-k)$ non-real zeros while if $(n-k)$ is odd, F has one real negative zero and $(n-k-1)$ non-real zeros.*

Proof. First we observe that since b is real, $F(-n, b; -2n; z)$ is a polynomial with real coefficients and therefore all non-real zeros of F must occur in conjugate pairs. From (2.1) we have

$$F(-n, b; -2n; z) = \frac{n!\, z^n}{(-2n)_n} P_n^{(\alpha, -\alpha)}\left(1 - \frac{2}{z}\right),$$

where $\alpha = -b - n$, so that $\alpha < 0$ corresponds to $b > -n$ and $n + \alpha = -b$. Therefore in (2.7)

$$X = E(1-b). \tag{3.1}$$

(i) Suppose $b > 0$. Then $1 - b < 1$ and it follows from (2.3) that $X = 0$. Also

$$\binom{n+\alpha}{n} = \binom{-b}{n} = (-1)^n\, b(b+1) \cdots (b+n-1)/n! \tag{3.2}$$

implies that

$$\operatorname{sgn} \binom{n+\alpha}{n} = \operatorname{sgn} \binom{-b}{n} = (-1)^n.$$

Therefore from (2.4) we have $N_1 = 0$. Also $N_2 = 0$ and

$$N_3 = \begin{cases} 0 & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Finally, $1 - \frac{2}{z} > 1$ implies $z < 0$ so that the one real zero of F that occurs when n is odd is negative.

(ii) Suppose that $-k < b < -k+1$, $k \in \{1, ..., n\}$. Then $k < 1-b < k+1$ so from (3.1) and (2.3) we have

$$X = E(1-b) = k, \qquad k = 1, ..., n.$$

From (3.2) it follows that if $-k < b < -k+1$, $\binom{n+\alpha}{n} = \binom{-b}{n}$ has sign equal to $(-1)^{n-k}$. Therefore, from (2.4),

$$N_1 = \begin{cases} 2[(k+1)/2] & \text{if } n \text{ is even} \\ 2[k/2]+1 & \text{if } n \text{ is odd} \end{cases}$$

and we see that $N_1 = k$ for all $k = 1, ..., n$. Since N_1 is the number of real zeros of $P_n^{(-b-n, b+n)}(w)$ in $(-1, 1)$, it follows from (2.1) that $F(-n, b; -2n; z)$ has k real zeros in $(1, \infty)$ for $-k < b < -k+1$, $k = 1, ..., n$. Further, $N_2 = 0$, whereas

$$N_3 = \begin{cases} 0 & \text{if } (n-k) \text{ even} \\ 1 & \text{if } (n-k) \text{ odd} \end{cases}$$

and this completes the proof. ▮

Corollary 3.2. *Let* $F(-n, b; -2n; z)$ *be defined by* (1.1) *with* b *real and* $n \in \mathbb{N}$.

(i) *For* $b < -2n$, *all* n *zeros of* F *are non-real for* n *even whereas for* n *odd*, F *has exactly one real zero in the interval* $(0, 1)$.

(ii) *For* $-n > b > -2n$, $-n-k > b > -n-k-1$, $k \in \{0, ..., n-1\}$, $F(-n, b; -2n; z)$ *has* $(n-k)$ *real zeros in the interval* $(1, \infty)$. *In addition, if* k *is even*, F *has* k *non-real zeros while if* k *is odd*, F *has one real zero in* $(0, 1)$ *and* $(k-1)$ *non-real zeros.*

Proof. In preference to the symmetry relation (2.2), we can use the identity (cf. [8, p. 454, Eq. (3)])

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right),$$

which yields, with $a=-n$ and $c=-2n$,

$$F(-n, b; -2n; z)=(1-z)^n F\left(-n, -b-2n; -2n; \frac{z}{z-1}\right). \tag{3.3}$$

(i) For $b<-2n$, we have $-b>2n$ and so $-b-2n>0$. Then Theorem 3.1(i) implies that for $-b-2n>0$, the n zeros of $F(-n, -b-2n; -2n; w)$ are all non-real if n is even whereas if n is odd, there are $(n-1)$ non-real zeros and one real negative zero. Now if $w=\frac{z}{z-1}$ is negative, then $z\in(0, 1)$ which proves (i).

(ii) For $-n>b>-2n$, if $-n-k>b>-n-k-1$, $k\in\{0, 1, ..., n-1\}$, then $-(n-k)<-2n-b<-(n-k)+1$. We can deduce from Theorem 3.1(ii) that $F(-n, -2n-b; -2n; w)$ has $(n-k)$ real zeros in the interval $(1, \infty)$. Now, if $w=\frac{z}{z-1}>1$, we have $z>1$ so that it follows from (3.3) that $F(-n, b; -2n; w)$ has $(n-k)$ real zeros for $-n-k>b>-n-k-1$, $k=0, ..., n-1$, in the interval $(1, \infty)$. Further, again from Theorem 3.1(ii), if k is even, F has k non-real zeros whereas if k is odd, F has $(k-1)$ non-real zeros and one real zero in the interval $(0, 1)$. ∎

4. ASYMPTOTICS OF THE ZEROS OF $F(-n, b; -2n; z)$ FOR $b>0$

THEOREM 4.1. *Let $b>0$ and $n\in\mathbb{N}$. For the zeros z of the hypergeometric polynomial $F(-n, b; -2n; z)$ we have that $1/z$ approaches the Cassini curve $|(2w-1)^2-1|=1$ as $n\to\infty$; more precisely, if Z_n is the set of the zeros of $F(-n, b; -2n; z)$, then*

$$\max_{z\in Z_n} \min\left\{\left|\frac{1}{z}-w\right| : w\in\mathbb{C}, |(2w-1)^2-1|=1\right\}\to 0 \qquad \textit{as} \quad n\to\infty.$$

PROPOSITION 4.2. *Let $b\in\mathbb{C}$ with $b\neq 0, -1, -2, ...$ and $n\in\mathbb{N}$. If z is a zero of the hypergeometric polynomial $F(-n, b; -2n; z)$, then*

$$u=\frac{4}{z}-\frac{4}{z^2} \tag{4.1}$$

is a zero of the hypergeometric function

$$_3F_2\left(b, -b-2n, -n; -2n, -n+\frac{1}{2}; \frac{1}{u}\right). \tag{4.2}$$

Proof. The quadratic transformation

$$F(a, b; 2b; z) = (1-z)^{-a/2} F\left(\frac{a}{2}, b-\frac{a}{2}; b+\frac{1}{2}; \frac{z^2}{4z-4}\right),$$

see [1, (15.3.15)], with $b=-n$, gives

$$F(a, -n; -2n; z) = (1-z)^{-a/2} F\left(\frac{a}{2}, -n-\frac{a}{2}; -n+\frac{1}{2}; \frac{z^2}{4z-4}\right).$$

With u as in (4.1) and noting that the order of the numerator parameters in F is immaterial, the last equation can be written

$$F(-n, b; -2n; z) = (1-z)^{-b/2} F\left(\frac{b}{2}, -n-\frac{b}{2}; -n+\frac{1}{2}; \frac{1}{u}\right). \tag{4.3}$$

A formula due to Clausen, see [10, (2.57)], states that

$$[F(c, b; c+b+\tfrac{1}{2}; z)]^2 = {}_3F_2(2c, 2b, c+b; 2c+2b, c+b+\tfrac{1}{2}; z),$$

and so it follows from (4.3) that

$$[F(-n, b; -2n; z)]^2 = (1-z)^{-b}\, {}_3F_2\left(b, -2n-b, -n; -2n, -n+\frac{1}{2}; \frac{1}{u}\right).$$

Since $z=1$ is not a zero of $F(b, -n; -2n; z)$ for $b \neq 0, -1, -2, \ldots$, the result follows. ▮

For simplicity of notation we write

$$u^n\, {}_3F_2\left(b, -b-2n, -n; -2n, -n+\frac{1}{2}; \frac{1}{u}\right) =: H_n(b, u)$$

$$=: \sum_{l=0}^{n} a_{n,l} u^{n-l}. \tag{4.4}$$

PROPOSITION 4.3. (i) *For all integers* $0 \leqslant l \leqslant n$ *we have* $a_{n,l} \leqslant e^{b(\gamma+1)}(l+1)^b$, *where* γ *is Euler's constant.*

(ii) *For all integers* $1 \leqslant l \leqslant n$ *we have* $a_{n,n-l}/a_{n,n} \leqslant e^{1/b-1+\gamma(1-b)} l^{1-b}$.

(iii) *If* $b \geqslant 1$, *then* $a_{n,l+1} > a_{n,l}$ *for* $0 \leqslant l < n$.

Proof. (i) We have

$$\frac{a_{n,l+1}}{a_{n,l}} = \frac{(2n-2l)(b+l)(b+2n-l)}{(2n-2l-1)(l+1)(2n-l)}. \tag{4.5}$$

Also,

$$\frac{2n-2l}{2n-2l-1}=1+\frac{1}{2n-2l-1},$$

$$\frac{b+l}{1+l}=1+\frac{b-1}{l+1}, \tag{4.6}$$

$$\frac{b+2n-l}{2n-l}=1+\frac{b}{2n-l},$$

implies that

$$\log\left(\frac{2n-2l}{2n-2l-1}\right)\leqslant\frac{1}{2n-2l-1},$$

$$\log\left(\frac{b+l}{1+l}\right)\leqslant\frac{b-1}{l+1},$$

$$\log\left(\frac{b+2n-l}{2n-l}\right)\leqslant\frac{b}{2n-l}.$$

Hence

$$\begin{aligned}\log a_{n,l}&=\log\left(\prod_{j=0}^{l-1}\frac{a_{n,j+1}}{a_{n,j}}\right)\\&\leqslant\sum_{j=0}^{l-1}\left(\frac{1}{2n-2j-1}+\frac{b-1}{j+1}+\frac{b}{2n-j}\right)\\&\leqslant\sum_{j=0}^{l-1}\frac{b}{j+1}+b\leqslant b(\log(l+1)+\gamma+1),\end{aligned}$$

which gives the stated estimate.

(ii) From (4.5) and (4.6) it follows immediately that

$$\frac{a_{n,l}}{a_{n,l+1}}\leqslant\frac{1+l}{b+l}=1+\frac{1-b}{b+l}.$$

Hence

$$\begin{aligned}\log\left(\frac{a_{n,\,n-l}}{a_{n,\,n}}\right) &= \log\left(\prod_{j=1}^{l} \frac{a_{n,\,n-j}}{a_{n,\,n-j+1}}\right)\\ &\leqslant \sum_{j=1}^{l} \log\left(1+\frac{1-b}{b+n-j}\right)\\ &\leqslant \sum_{j=1}^{l} \frac{1-b}{b+j}\\ &\leqslant (1-b)\left(\frac{1}{b}+\log(l)+\gamma\right).\end{aligned}$$

(iii) This immediately follows from the fact that all terms in (4.6) are $\geqslant 1$ and at least two of them are >1. ▮

PROPOSITION 4.4. *Let $b>0$. For all $n \in \mathbb{N}$ there are numbers $0<\rho_n<1<r_n$ with $\rho_n \to 1$ and $r_n \to 1$ as $n \to \infty$ such that all zeros of $H_n(b, u)$ lie in the annulus $\{u \in \mathbb{C} : \rho_n \leqslant |u| \leqslant r_n\}$.*

Proof. Let $0<\rho<1$. From Proposition 4.3 it immediately follows that the sequence of polynomials $u^n(H_n(b, \frac{1}{u}))_{n=2}^{\infty}$ is uniformly bounded on $\{u \in \mathbb{C} : |u|<\rho\}$. Moreover, since for fixed l,

$$a_{n,\,l} = \frac{(2n-2l+1)_l\,(-b-2n)_l}{(1/2-n)_l^2\,(-4)^l}\;\frac{(b)_l}{(1)_l} \to \frac{(b)_l}{(1)_l} \qquad \text{as} \quad n \to \infty,$$

$u^n H_n(b, \frac{1}{u})$ converges pointwise, and therefore uniformly by Vitali's theorem, to

$$\sum_{l=0}^{\infty} \frac{(b)_l}{(1)_l}\, u^l = F(b, 1; 1; u) = (1-u)^{-b}.$$

Since the latter function does not have any zeros inside the unit disk, by Hurwitz's theorem there is an index n_0 such that also $u^n H_n(b, \frac{1}{u})$ does not have zeros in $\{u \in \mathbb{C} : |u|<\rho\}$ for $n>n_0$, i.e., $H_n(b, u)$ does not have zeros $|u|>\frac{1}{\rho}$, and we can ensure that $r_n \leqslant \frac{1}{\rho}$ for $n \geqslant n_0$.

Now we consider

$$a_{n,\,n}^{-1} H_n(b, u) = \sum_{l=0}^{n} \frac{a_{n,\,n-l}}{a_{n,\,n}}\, u^l.$$

From Proposition 4.3 we infer that $a_{n,n}^{-1} H_n(b, u)$ is uniformly bounded on $\{u \in \mathbb{C} : |u| < \rho\}$. Moreover, since for fixed l,

$$\frac{a_{n,n-l}}{a_{n,n}} = \frac{(1/2-l)_l^2 \, (-4)^l \, (1)_{n+1}}{(-b-n-l)_l \, (b+n-l)_l \, (1)_{2l} \, (1)_{n-l}}$$

$$\to \frac{(1/2-l)_l^2 \, (4)^l}{(1)_{2l}} = \frac{(1/2)_l}{(1)_l} \qquad \text{as} \quad n \to \infty,$$

$a_{n,n}^{-1} H_n(b, u)$ converges pointwise, and therefore uniformly by Vitali's theorem, to

$$\sum_{l=0}^{\infty} \frac{(1/2)_l}{(1)_l} u^l = F\left(\frac{1}{2}, 1; 1; u\right) = (1-u)^{-1/2}.$$

Since the latter function does not have any zeros inside the unit disk, by Hurwitz's theorem there is an index n_0 such that also $H_n(b, u)$ does not have zeros in $\{u \in \mathbb{C} : |u| < \rho\}$ for $n > n_0$, and we can ensure that $\rho_n \geqslant \rho$ for $n \geqslant n_0$. ∎

Proof of Theorem 4.1. Theorem 4.1 immediately follows from Proposition 4.4 if we observe that

$$\left(\frac{1}{2z} - 1\right)^2 - 1 = \frac{1}{4z^2} - \frac{1}{4z} = -u. \quad ∎$$

5. GRAPHS AND ADDITIONAL REMARKS

For the graphs it is more convenient to use the independent variable $w = \frac{1}{z}$ since then the limit curve of the zeros is a Cassini curve. Together with the notations in the previous section this leads to

$$(2w-1)^2 - 1 = -u. \tag{5.1}$$

From (5.1) we see that every value of u leads to two values for w, one with $\Re w \geqslant \frac{1}{2}$ and one with $\Re w \leqslant \frac{1}{2}$. However, evidence from plots of the zeros suggests that always $\Re w < \frac{1}{2}$, and hence we conjecture that there is a one-to-one correspondence between zeros z of $F(-n, b; -2n; z)$ and the zeros of $H_n(b, u)$, i.e., z would be a zero of $F(-n, b; -2n; z)$ if and only if $\Re \frac{1}{z} < \frac{1}{2}$ and $u = \frac{4}{z} - \frac{4}{z^2}$ is a zero of $H_n(b, u)$. In particular, all zeros of $F(-n, b; -2n; 1/w)$ would approximate the left branch of the Cassini curve $|(2w-1)^2 - 1| = 1$ as $n \to \infty$.

We have been unable to prove the above conjecture; however, we have the following statement that this is true at least asymptotically, more precisely:

PROPOSITION 5.1. *Let* $b>0$ *and* $n\in\mathbb{N}$. *For every* $\varepsilon>0$ *there is* $N\in\mathbb{N}$ *such that for* $n\geqslant N$ *all zeros* z *of the hypergeometric polynomial* $F(-n, b; -2n; z)$ *satisfy* $\Re\frac{1}{z}<\frac{1}{2}+\varepsilon$.

Proof. Observe that $\Re\frac{1}{z}>\frac{1}{2}$ if and only if $|1-z|<1$. From the linear relation between hypergeometric functions

$$F(-n, b; -2n; z)=\frac{(-2n-b)_n}{(-2n)_b}F(-n, b; 1+n+b; 1-z),$$

see, e.g., [4], the statement of the theorem will follow with a reasoning as in the proof of Proposition 4.4. As there it can be shown that, uniformly on compact subsets of the open unit disk,

$$F(-n, b; 1+n+b; z)=1+\sum_{k=1}^{n}\frac{(n-k+1)_k\,(b)_k}{(1+n+b)_k}\frac{(-z)^k}{k!}$$

converges to $(1+z)^b$, and a reasoning as in Proposition 4.4 completes the proof. ∎

The following result shows that for $b\geqslant 1$ the zeros of $w^nF(-n, b; -2n; 1/w)$ lie outside the limit curve of the zeros.

PROPOSITION 5.2. *For* $b\geqslant 1$, *all zeros* w *of* $w^nF(-n, b; -2n; 1/w)$ *lie outside the Cassini curve* $|(\zeta-1)^2-1|=1$, *i.e., satisfy* $|(w-1)^2-1|>1$.

Proof. From Proposition 4.4(iii) and the Eneström–Kakeya Theorem (cf. [5, p. 136]) we see that all zeros of $H_n(b, u)$ lie outside the unit disk, which immediately gives the statement. ∎

Although we have shown that the zeros of $F(-n, b; -2n; z)$ have a certain asymptotic behaviour, this is only a first approximation. Indeed, the graphs in Fig. 1 indicate that the zeros w of $F(-n, b; -2n; 1/w)$ are very close to Cassini curves

$$|(w-d)^2-c^2|=a^2$$

with suitable real numbers a, c, d.

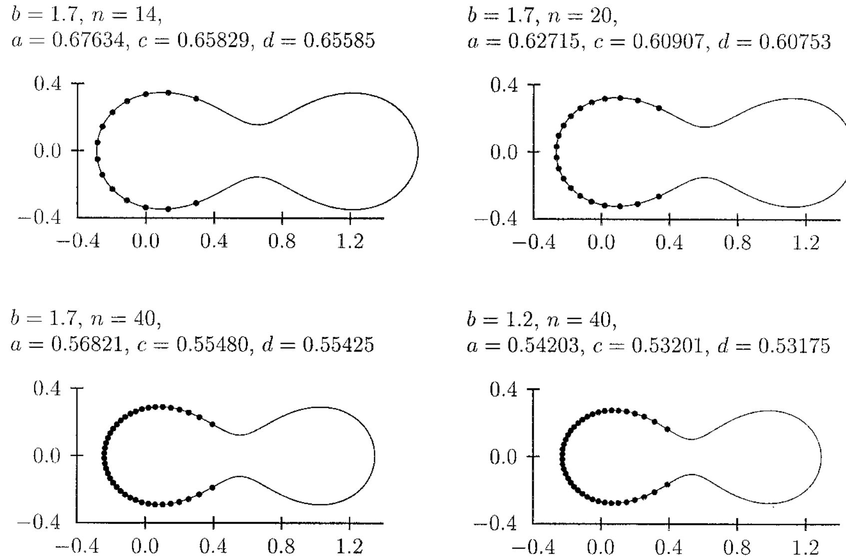

FIGURE 1

We conjecture that the zeros of $H_n(b, u)$ "approach" the $(n+1)$st unit roots different from 1 as $n \to \infty$, more precisely, we can enumerate the zeros $a_{n,k}$, $k = 1, ..., n$, of (4.2) in such a way that

$$a_{n,k} = r_{n,k} \exp\left(\frac{2k\pi i}{n} \alpha_{n,k}\right),$$

where $r_{n,k} > 0$ and $\alpha_{n,k}$ are real numbers such that

$$r_{n,k} = 1 + o(1/n), \qquad \alpha_{n,k} = 1 + o(1/n).$$

In Theorem 4.1 we have indeed shown that $r_{n,k} = 1 + o(1/n)$, and we should find a better approximation, say

$$r_{n,k} = 1 + \rho_{n,k} + \varepsilon_{n,k},$$

with explicitly given $\rho_{n,k}$ and "very small" $\varepsilon_{n,k}$. However, our method does not provide such a better estimate, and it gives no information at all about the argument of the roots.

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